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Combinatorial deformations of theta series of various kinds

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1 Introduction

It is known that there are strong links between combinatorcs and number theory. From self-dual binary or ternary (or other kinds of) codes we obtain unimodular lattices. From various kinds of weight enumerators we can define theta series of various kinds. However for the present speaker there is one unsatisfactory point, which will be precisely desribed in the former part of the present talk.

In the latter part we propose some possible remedies.

2 Some preliminaries

2.1 Binary self-dual codes

Let $\mathbf{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbf{F}_2^n$ be the vector space of dimension n over $\mathbf{F}_2$. A linear $[n, k]$ code $\mathbf{C}$ is a vector subspace of V of dimension k. In V , the inner product , which is denoted by $(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y}$ in V, is defined as usual. The dual code $\mathbf{C}^{\perp}$ of C is defined by

$$\mathbf{C}^{\perp} = \{\mathbf{u} \in V \mid (\mathbf{u}, \mathbf{v}) = \mathbf{0} \quad \forall \mathbf{v} \in \mathbf{C}\}.$$

A code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^{\perp}$.

An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. Let $\mathbf{x}$ be a codeword of a linear $[n, k]$ code $\mathbf{C}$, then the Hamming weight $wt(\mathbf{x})$ of the codeword

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

is defined to be the number of i's such that $x_i \neq 0$. The Hamming distance d on $\mathbf{C}$ is also defined by $d(\mathbf{x},\mathbf{y}) = wt(\mathbf{x}-\mathbf{y})$.
Let $\mathbf{C}$ be a self-dual binary $[n, \frac{n}{2}]$ code, then the weight $wt(\mathbf{x})$ of each codeword $\mathbf{x}$ in $\mathbf{C}$ is an even number. Further, if the weight of each codeword $\mathbf{x}$ in $\mathbf{C}$ is divisible by 4, then the code is called a doubly even binary code. It is known that doubly even self-dual binary codes $\mathbf{C}$ exist only when the length n of $\mathbf{C}$ is a multiple of 8.

The minimum distance $d(\mathbf{C})$ for a code $\mathbf{C}$ is defined by

$$d(\mathbf{C}) = \min_{\mathbf{u},\mathbf{v}\in\mathbf{C},\mathbf{u}\neq\mathbf{v}} d(\mathbf{u},\mathbf{v}) = \min_{\mathbf{u}\in\mathbf{C}-\{\mathbf{0}\}} wt(\mathbf{u}).$$

There is a well known proposition :

Proposition 1 *Let* $\mathbf{C}$ *be a doubly even self dual binary code of length* n*, then*

$$d(\mathbf{C}) \leq 4\left[\frac{n}{24}\right] + 4.$$

Definition : A doubly even self dual binary code of length n satisfying

$$d(\mathbf{C}) = 4\left[\frac{n}{24}\right] + 4$$

is called an extremal code.

Let $\mathbf{C}$ be a self-dual doubly even code of length n, which is embedded in $\mathbf{F}_2^n$. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n)$ be any pair of vectors in $\mathbf{F}_2^n$, then the number of common 1's of the corresponding coordinates for $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} * \mathbf{v}$. This is called the intersection number of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} * \mathbf{u}$ is nothing else $wt(\mathbf{u})$.

Let $\mathbf{C}$ be a doubly even self-dual binary $[n, \frac{n}{2}]$ code. The homogeneous weight enumerator $W_{\mathbf{C}}(x,y)$ of the code $\mathbf{C}$ is defined by

$$W_{\mathbf{C}}(x,y) = \sum_{\mathbf{v}\in\mathbf{C}} x^{n-wt(\mathbf{v})} y^{wt(\mathbf{v})}.$$

A basic result is the MacWilliams identity for binary self-dual code :

Theorem 1 *Let* $W_{\mathbf{C}}(x,y)$ *be the weight enumerator of a self-dual binary* $[n, \frac{n}{2}]$ *code, then the following identity holds :*

$$W_{\mathbf{C}}(x,y) = W_{\mathbf{C}}(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}). \qquad (1)$$

When $\mathbf{C}$ is a doubly even binary code then $W_{\mathbf{C}}(x,y)$ has another inbariance property :

$$W_{\mathbf{C}}(x,iy) = W_{\mathbf{C}}(x,y) \qquad (2)$$

Definition of Jacobi polynomials for code

The homogeneous Jacobi polynomial $Jac(\mathbf{C},\mathbf{v};x,y,u,v)$ for $\mathbf{C}$ with respect to $\mathbf{v} \in \mathbf{F}_2^n$ is defined by

$$\begin{aligned} &Jac(\mathbf{C},\mathbf{v};x,y,u,v) \\ &= \sum_{\mathbf{u}\in\mathbf{C}} x^{n-wt(\mathbf{v})-wt(\mathbf{u})+\mathbf{u}*\mathbf{v}} y^{wt(\mathbf{u})-\mathbf{u}*\mathbf{v}} u^{wt(\mathbf{v})-\mathbf{u}*\mathbf{v}} v^{\mathbf{u}*\mathbf{v}} \end{aligned}$$

We will call the vector $\mathbf{v}$ the reference vector of the Jacobi polynomial in some occasions. The weight $wt(\mathbf{v})$ in the polynomial $Jac(\mathbf{C},\mathbf{v};x,y,u,v)$ is called the index of the polynomial.

We get a MacWilliams type identity for $Jac(\mathbf{C},\mathbf{v};x,y,u,v)$

Theorem 2 *Let $W_{\mathbf{C}}(x,y)$ be the weight enumerator of a doubly even self-dual binary $[n,\frac{n}{2}]$ code, then the following identity holds :*

$$Jac(\mathbf{C},\mathbf{v};x,y,u,v) = Jac(\mathbf{C},\mathbf{v};\frac{x+y}{\sqrt{2}},\frac{x-y}{\sqrt{2}},\frac{u+v}{\sqrt{2}},\frac{u-v}{\sqrt{2}}). \tag{3}$$

Since $\mathbf{C}$ is doubly even, each codeword $\mathbf{u}$ of $\mathbf{C}$ has weight divisible by 4, and we get (C.f. [13])

$$Jac(\mathbf{C},\mathbf{v};x,iy,u,iv) = Jac(\mathbf{C},\mathbf{v};x,y,u,v) \tag{4}$$

2.2 Certain finite groups and their invarint rings

Let G_1 be the group generated by

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

This group is of order 192 and is known as No.9 in Shephard-Todd's list in [20]. The equations (1) and (2) show that the homogeneous weight enumerator $W_{\mathbf{C}}(x,y)$ for a doubly even self-dual binary code belongs to the ring of invariant polynomials $\mathbb{C}[x,y]^{G_1}$ for the finite group of linear transformations G_1. Let $\mathcal{W}[x,y]$ be the subring of $\mathbb{C}[x,y]$ generated by the homogeneous weight enumerators of doubly even self-dual binary codes.
Then Gleason's theorem can be regarded that the ring $\mathcal{W}[x,y]$ coincides with the ring $\mathbb{C}[x,y]^{G_1}$.
We denote by $diag(G_1,G_1)$ the group of linear transformations generated by

$$\tilde{\sigma_1} = \begin{pmatrix} \sigma_1 & O \\ O & \sigma_1 \end{pmatrix} \text{and } \tilde{\sigma_2} = \begin{pmatrix} \sigma_2 & O \\ O & \sigma_2 \end{pmatrix}.$$

Then the equations (3) and (4) show that the polynomial $Jac(\mathbf{C},\mathbf{v};x,y,u,v)$ for doubly even self-dual binary code $\mathbf{C}$ is a polynomial in the ring $\mathbb{C}[x,y,u,v]^{diag(G_1,G_1)}$ of simultaneous polynomial invariants for the group G_1 in the sense of I. Schur [19] pages 9-14.

2.3 Construction of lattices from binary codes

Construction A.

Let $\mathbf{C}$ be a doubly even self-dual binary code of length n. ($n \equiv 0 \pmod 8$) Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be an element of $\mathbb{Z}^n$, where $\mathbb{Z}$ is the ring of rational integers. A map ρ is defined by

$$\begin{array}{rcl} \rho : \mathbb{Z}^n & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^n = \mathbf{F}_2^n \\ \Downarrow & & \Downarrow \\ \mathbf{x} & \longmapsto & \mathbf{x} \bmod 2 \end{array}$$

The set $L(\mathbf{C})$ defined by

$$L(\mathbf{C}) = \frac{1}{\sqrt{2}}\rho^{-1}(\mathbf{C})$$

is proved to be an even unimodular lattice of dimension n. This lattice construction is called the Construction A.

Construction B.

Let $\mathbf{C}$ be a doubly even self-dual binary code of length n. The set defined by

$$\mathcal{L}(\mathbf{C}) = \left\{ \frac{1}{\sqrt{2}}\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n,\ \mathbf{x}\ mod2 \in \mathbf{C},\ \sum_{i=1}^{n} x_i \equiv 0 \pmod 4 \right\}$$

is an even lattice. This lattice construction is called the Construction B.

2.4 Modular forms, Jacobi forms

2.4.1 Definition of modular forms

Let $\mathbb{H}$ be the complex upper half plane. Let τ be a variable on $\mathfrak{H}$. A holomorphic function $f(\tau)$ on $\mathbb{H}$ is called a modular form of weight k (k is even) with respect $SL_2(\mathbb{Z})$ if it satisfies following conditions (5) and (6) :

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \quad \text{holds for} \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{5}$$

A special case of (5) implies that $f(\tau)$ satisfies $f(\tau+1) = f(\tau)$, and $f(\tau)$ has a Fourier expansion :

$$f(\tau) = \sum_{m\in\mathbb{Z}} a_m e^{2\pi i m\tau}.$$

$$f(\tau) = \sum_{m\geq 0, m\in\mathbb{Z}} a_m e^{2\pi i m\tau}. \tag{6}$$

The set of modular forms of weight k with respect to $SL_2(\mathbb{Z})$ is denoted by $M(k)$.

2.4.2 Definition of Jacobi forms

Let $\mathbb{H}$ and τ be as above. Let $\mathbb{C}$ be the complex plane and z be a variable on $\mathbb{C}$. A complex valued holomorphic function $\phi(\tau, z)$ defined on $\mathbb{H} \times \mathbb{C}$ is called a Jacobi form of weight k and index h with respect to the pair ($SL_2(\mathbb{Z})$,$\mathbb{Z}$) if it satisfies the conditions (7), (8) and (9) :

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \phi(\tau, z) = (c\tau+d)^{-k} e^{2\pi i h\left(\frac{-cz^2}{c\tau+d}\right)} \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \tag{7}$$

$$\phi(\tau, z) = e^{2\pi i h(\lambda^2\tau + 2\lambda z)} \phi(\tau, z+\lambda\tau+\mu) \tag{8}$$

$\phi(\tau, z)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n\geq r^2/4h} c(n, r) q^n \zeta^r \tag{9}$$

The set of Jacobi forms of weight k and index h with respect to the pair $(SL_2(\mathbb{Z},\mathbb{Z})$ is denote by $\mathcal{J}(k,h)$.

2.5 Theta series, Jacobi theta series

Let L be an even unimodular lattice of rank n. (n is divisible by 8) Theta series attached t the lattice is defined by

$$\vartheta(\tau, L) = \sum_{\mathbf{x}\in L} e^{\pi i(\mathbf{x},\mathbf{x})\tau}.$$

It is known that $\vartheta(\tau, L)$ is a modular form of weight $n/2$.

Let $\mathbf{y} \in L$ satisfying $(\mathbf{y},\mathbf{y}) = 2h$, then a Jacobi theta series with respect to $\mathbf{y}$ is defined by

$$\vartheta_{\mathbf{y}}(\tau, z, L) = \sum_{\mathbf{x}\in L} e^{\pi i(\mathbf{x},\mathbf{x})\tau + 2\pi i(\mathbf{x},\mathbf{y})z}.$$

This series is proved to be a Jacobi form of weight $n/2$ and index h.

2.6 Jacobi's theta functions

Let ϵ, ϵ' be numbers 0 or 1. Then Jacobi's theta functions are defined by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau, z) = \sum_{h\in\mathbb{Z}} e^{\pi i\{\tau(h+\epsilon/2)^2 + 2(h+\epsilon/2)(z+\epsilon'/2)\}}$$

A more popular notations are the right-hand sides of the followings

$$\begin{aligned}
\theta\begin{bmatrix}0\\0\end{bmatrix}(\tau,z) &= \theta_3(\tau,z)\\
\theta\begin{bmatrix}0\\1\end{bmatrix}(\tau,z) &= \theta_0(\tau,z)\\
\theta\begin{bmatrix}1\\0\end{bmatrix}(\tau,z) &= \theta_2(\tau,z)\\
\theta\begin{bmatrix}1\\1\end{bmatrix}(\tau,z) &= \theta_1(\tau,z)
\end{aligned}$$

These functions satisfy the following transformations :(Conf. [17],[23].)

$$\theta_2(-\frac{1}{\tau}, \frac{z}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \frac{z^2}{\tau}} \theta_0(\tau, z) \tag{10}$$

$$\theta_0(-\frac{1}{\tau}, \frac{z}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \frac{z^2}{\tau}} \theta_2(\tau, z) \tag{11}$$

$$\theta_3(-\frac{1}{\tau}, \frac{z}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \frac{z^2}{\tau}} \theta_3(\tau, z) \tag{12}$$

here we take the branch of the root sothat it takes the value 1 at $\tau = i$.

If we put

$$\varphi_0(\tau,z)=\theta_0(2\tau,2z),\ \varphi_2(\tau,z)=\theta_2(2\tau,2z),\ \varphi_3(\tau,z)=\theta_3(2\tau,2z)$$

$$\varphi_0(\tau)=\theta_0(2\tau,0),\varphi_2(\tau)=\theta_2(2\tau,0),\varphi_3(\tau)=\theta_3(2\tau,0)$$

then we can show that

$$\varphi_2(-\frac{1}{\tau},\frac{z}{\tau})=\sqrt{\frac{\tau}{2i}}e^{\pi i\frac{z^2}{\tau}}\{\varphi_3(\tau,z)-\varphi_2(\tau,z)\}$$

$$\varphi_3(-\frac{1}{\tau},\frac{z}{\tau})=\sqrt{\frac{\tau}{2i}}e^{\pi i\frac{z^2}{\tau}}\{\varphi_3(\tau,z)+\varphi_2(\tau,z)\}$$

holds.

3 Statement of the Problem

Problem 1.

In what way the extremal theta series (i.e. theta series for extremal lattices) of various kinds (ordinary, Jacobi, Siegel) can be determined ? Is there a clear way that leads to direct construction of extremal theta series ? At present before us there is a vague and ugly indirect way that leads to such construction.

Here we give a precise explanation of the problem. Throughout this section we let **C** be a doubly even self-dual binary code of length n.

3.1 Ordinary theta series case

Let $W_{\mathbf{C}}(x,y)$ be the weight enumerator of the code **C**, then the Broué-Enguehard map is a correspondence

$$\begin{array}{ccc}(\mathbb{C}[x,y]^{\mathbf{G}_1})_h & \longrightarrow & M_{n/2}\\ \Downarrow & & \Downarrow\\ W_{\mathbf{C}}(x,y) & \longmapsto & W_{\mathbf{C}}(\varphi_3(\tau),\varphi_2(\tau))\end{array}$$

Here $(\mathbb{C}[x,y]^{\mathbf{G}_1})_h$ is the vector space of the h-th homogeneous part of $\mathbb{C}[x,y]^{\mathbf{G}_1}$.
They proved this by showing that $W_{\mathbf{C}}(\varphi_3(\tau),\varphi_2(\tau))$ is a theta series $\vartheta(\tau,L(\mathbf{C}))$ of the even unimodular lattice $L(\mathbf{C})$.
The problem is that even if **C** is an extremal code of length n the resulting theta series $\vartheta(\tau,L(\mathbf{C}))$ is not the extremal theta series. For instance when $\mathcal{G}_{24}$ is the binary Golay code of length 24, then $\vartheta(\tau,L(\mathcal{G}_{24}))$ is not the extremal theta series, but it is the theta series associated with the even unimodular lattice of type A_1^{24} (coming from the root lattice of A_1^{24}). How the theta series of the Leech lattice can be derived ?

Numerical evidence 1.

$$W_{\mathcal{G}_{24}}(x,y)=x^{24}+759x^{16}y^8+2576x^{12}y^{12}+759x^8y^{16}+y^{24}$$

is the weight enumerator of the binary Golay $[24, 12, 8]$ code.

$$
\begin{aligned}
&W_{\mathcal{G}_{24}}(\varphi_3(\tau), \varphi_2(\tau)) = \\
&\vartheta(\tau, L(\mathcal{G}_{24})) = \\
&\quad 1 + 48q + 195408q^2 + 16785216q^3 + 397963344q^4 + 4629612960q^5 + 34417365696q^6 + \cdots,
\end{aligned}
$$

here $q = e^{2\pi i \tau}$. On the other hands theta series for Leech lattice is

$$
\begin{aligned}
&\vartheta(\tau, Leech) = \\
&\quad 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + 4629381120q^5 + 34417656000q^6 + \cdots
\end{aligned}
$$

This fact compels us to pose the following question :
Can we modify the Broué-Enguehard map to the effect that the following diagram is commutative

$$
\begin{array}{ccl}
\mathbb{C}[x, y]^{\mathbf{G}_1} & \longrightarrow & M_{n/2} \ni \vartheta(\tau, L(\mathbf{C})) \\
\downarrow & & \downarrow \\
\mathbb{C}[x, y]^{\mathbf{G}_1} & \longmapsto & M_{n/2} \ni \text{extremal theta}
\end{array}
$$

3.2 Jacobi theta series case

Let $Jac(\mathbf{C}, \mathbf{v}, x, y, u, v)$ be a Jacobi polynomial for the code $\mathbf{C}$ with a reference vector $\mathbf{v}$ of weight 1, the (so called and the simplest) Bannai-Ozeki map is a correspondence

$$
\begin{array}{rcl}
(\mathbb{C}[x, y, u, v]^{diag(\mathbf{G}_1, \mathbf{G}_1)})_1 & \longrightarrow & J_1(n/2) \\
\Cup & & \Cup \\
Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) & \longmapsto & Jac(\mathbf{C}, \mathbf{v}; \varphi_3(\tau), \varphi_2(\tau), \varphi_3(\tau, z), \varphi_2(\tau, z))
\end{array}
$$

Numerical evidence 2.

$$
\begin{aligned}
&Jac(\mathcal{G}_{24}, \mathbf{v}, x, y, u, v) = \\
&\quad x^{23}u + 253x^{16}y^7v + 506x^{15}y^8u + 1288x^{12}y^{11}v + 1288x^{11}y^{12}u + 506x^8y^{15}v + \\
&\quad 253x^7y^{16}u + y^{23}v
\end{aligned}
$$

is the Jacobi polynomial of index 1 for the Golay code. The image of Bannai-Ozeki map for this polynomial is computed to be

$$
Jac(\mathcal{G}_{24}, \mathbf{v}; \varphi_3(\tau), \varphi_2(\tau), \varphi_3(\tau, z), \varphi_2(\tau, z)) =
$$

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$$
\begin{aligned}
&1+(\zeta^{-2}+46+\zeta^{2})q+\{46(\zeta^{2}+\zeta^{-2})+32384(\zeta+\zeta^{-1})+130548\}q^{2}\\
&+\{130548(\zeta^{2}+\zeta^{-2})+9175896+3674112(\zeta+\zeta^{-1})\}q^{3}\\
&+\Big\{\zeta^{4}+\zeta^{-4}+32384(\zeta^{3}+\zeta^{-3})+9175896(\zeta^{2}+\zeta^{-2})\\
&+95659392(\zeta+\zeta^{-1})+188227998\Big\}\,q^{4}\\
&+\Big(46(\zeta^{4}+\zeta^{-4})+3674112(\zeta^{-3}+\zeta^{3})+188227998(\zeta^{2}+\zeta^{-2})\\
&+1143025664(\zeta+\zeta^{-1})+1959757320)q^{5}\\
&+\Big\{130548(\zeta^{4}+\zeta^{-4})+95659392(\zeta^{3}+\zeta^{-3})+1959757320(\zeta^{2}+\zeta^{-2})\\
&+8506630272(\zeta+\zeta^{-1})+13293010632\Big\}\,q^{6}+\cdots,
\end{aligned}
$$

where $\zeta = e^{2\pi i z}$.
This is verified to be the Jacobi theta series of index 1 associated with the even unimodular lattice of type A_1^{24}.
Jacobi theta series of index 1 associated with Leech lattice is 0 by its meaning.

Numerical evidence 3.

There is a unique Jacobi polynomial $Jac(\mathcal{G}_{24}, \mathrm{v}, x, y, u, v)$ of index 2 for the Golay code

$$
\begin{aligned}
&Jac(\mathcal{G}_{24}, \mathrm{v}, x, y, u, v) =\\
&\quad x^{22}u^{2}+y^{22}v^{2}+352x^{15}y^{7}uv+1344x^{11}y^{11}uv+352x^{7}y^{15}uv+77x^{16}y^{6}v^{2}\\
&\quad +330x^{14}y^{8}u^{2}+616x^{12}y^{10}v^{2}+616x^{10}y^{12}u^{2}+330x^{8}y^{14}v^{2}+77x^{6}y^{16}u^{2}
\end{aligned}
$$

From it we obtain

$$
\begin{aligned}
&Jac(\mathcal{G}_{24}, \mathrm{v}; \varphi_3(\tau), \varphi_2(\tau), \varphi_3(\tau,z), \varphi_2(\tau,z))) =\\
&\quad 1+(2\zeta^{-2}+2\zeta^{2}+44)q+\Big(\zeta^{4}+\zeta^{-4}+5016(\zeta^{2}+\zeta^{-2})+45056(\zeta+\zeta^{-1})+95262\Big)\,q^{2}\\
&\quad +\Big\{44(\zeta^{4}+\zeta^{-4})+45056(\zeta^{3}+\zeta^{-3})+959288(\zeta^{2}+\zeta^{-2})+4149248(\zeta+\zeta^{-1})+6477944\Big\}\,q^{3}\\
&\quad +\Big\{95262(\zeta^{4}+\zeta^{-4})+4149248(\zeta^{3}+\zeta^{-3})+95432704(\zeta+\zeta^{-1})+32752192(\zeta^{2}+\zeta^{-2})+13310453\\
&\quad +\Big\{2(\zeta^{6}+\zeta^{-6})+45056(\zeta^{5}+\zeta^{-5})+6477944(\zeta^{4}+\zeta^{-4})+95432704(\zeta^{3}+\zeta^{-3})\\
&\quad +458048186(\zeta^{2}+\zeta^{-2})+1062150144(\zeta+\zeta^{-1})+1385304888\Big\}\,q^{5}\\
&\quad +\Big\{5016(\zeta^{6}+\zeta^{-6})+4149248(\zeta^{5}+\zeta^{-5})+133104532(\zeta^{4}+\zeta^{-4})+1062150144(\zeta^{3}+\zeta^{-3})\\
&\quad +3772360200(\zeta^{2}+\zeta^{-2})+7534989312(\zeta+\zeta^{-1})+9403848792\Big\}\,q^{6}+\cdots
\end{aligned}
$$

This series is verified to be Jacobi theta series of index 2 associated with the even unimodular lattice of type A_1^{24}
Jacobi theta series of index 2 associated with the Leech lattice is computed to be

$$\vartheta_{\mathbf{y}}(\tau, z, Leech) =$$

$$
\begin{aligned}
&1+\left\{\zeta^4+\zeta^{-4}+4600(\zeta^2+\zeta^{-2})+47104(\zeta+\zeta^{-1})+93150\right\}q^2\\
&+\left\{47104(\zeta^3+\zeta^{-3})+953856(\zeta^2+\zeta^{-2})+4147200(\zeta+\zeta^{-1})+6476800\right\}q^3\\
&+\left\{93150(\zeta^4+\zeta^{-4})+4147200(\zeta^3+\zeta^{-3})+32788800(\zeta^2+\zeta^{-2})\right.\\
&\left.+95385600(\zeta+\zeta^{-1})+133204500\right\}q^4\\
&+\left\{47104(\zeta^5+\zeta^{-5})+6476800(\zeta^4+\zeta^{-4})+95385600(\zeta^3+\zeta^{-3})\right.\\
&\left.+458086400(\zeta^2+\zeta^{-2})+1062195200(\zeta+\zeta^{-1})+1384998912\right\}q^5\\
&+\left\{4600(\zeta^6+\zeta^{-6})+4147200(\zeta^5+\zeta^{-5})+133204500(\zeta^4+\zeta^{-4})+1062195200(\zeta^3+\zeta^{-3})\right.\\
&\left.+3771829800(\zeta^2+\zeta^{-2})+7535462400(z+\zeta^{-1})+9403968600\right\}q^6+\cdots
\end{aligned}
$$

3.3 Siegel theta series case

We do not give a numerical example, but it is evident that W. Duke's correspondence from the multiple weight enumerator of a class of doubly even self-dual binary codes $\mathbf{C}$ of length n to Sielgel modular forms via Siegel theta series for $L(\mathbf{C})$ uses Construction A. And this should cause the obtained Siegel theta series are associated with lattices of type A_1^n (in general case) even if we use extremal codes.

4 First Approach

4.1 Ordinary theta series case

Our first approach for a solution to our problem is to find a polynomial which leads to theta series of the Leech lattice via Broué-Enguehard map. It is easy to find such a polynomial, but it may be difficult to give a deeper meaning to the polynomial. The polynomial in question is given by

$$
\begin{aligned}
&\widehat{W_{\mathcal{G}_{24}}}(x,y)\\
&= x^{24}+759x^{16}y^8+2576x^{12}y^{12}+759x^8y^{16}+y^{24}\\
&\quad -3(x^{20}y^4-4x^{16}y^8+6x^{12}y^{12}-4x^8y^{16}+x^4y^{20})\\
&= x^{24}-3x^{20}y^4+771x^{16}y^8+2558x^{12}y^{12}+771x^8y^{16}-3x^4y^{20}+y^{24}
\end{aligned}
$$

We verify that

$$\widehat{W_{\mathcal{G}_{24}}}(\varphi_3(\tau),\varphi_2(\tau))=\vartheta(\tau,Leech)$$

holds.

There are similar facts in dimensions 32 and 40.

4.2 Jacobi theta series case

In this case we look for a polynomial in which satisfies

$$\widehat{Jac}(\mathcal{G}_{24}, \mathbf{v}; \varphi_3(\tau), \varphi_2(\tau), \varphi_3(\tau,z), \varphi_2(\tau,z) = \vartheta_{\mathbf{y}}(\tau, z, Leech) \text{ with } (\mathbf{y},\mathbf{y}) = 4$$

$$\begin{aligned}
&\widehat{Jac}(\mathcal{G}_{24}, \mathbf{v}, x, y, u, v) \\
&= x^{22}u^2 + y^{22}v^2 + 368x^{15}y^7uv + 1312x^{11}y^{11}uv + 368x^7y^{15}uv - 1/2x^{20}y^2v^2 \\
&\quad -5/2x^{18}u^2y^4 + 73x^{16}y^6v^2 + 330x^{14}y^8u^2 + 623x^{12}y^10v^2 + 623x^{10}y^{12}u^2 \\
&\quad +330x^8y^{14}v^2 + 73x^6y^{16}u^2 - 5/2x^4y^{18}v^2 - 1/2y^{20}u^2x^2
\end{aligned}$$

5 Second Approach

In this section we discuss an arithmetical deformation of theta series of even unimodular lattice. At present the research is not fully explored, but we expect this approach will be fruitful.

We explain our idea by examining a special case.

Let $\mathcal{L}_1$ be an even unimodular lattice of dimension 24 with its root sublattice of type A_1^{24}. Here we give a presentation of the lattice $\mathcal{L}_1$. Let $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_{24}$ be mutually orthogonal vectors in $\mathbb{R}^{24}$ satisfying $(\mathbf{f}_i, \mathbf{f}_i) = 2$, $i = 1, 2, \cdots, 24$ (the coupling denotes the inner product). $\mathcal{L}_1$ is generated by the vectors $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_{24}$ and the vecorts of the form

$$\frac{1}{2}\sum_i^{24} \rho_i \mathbf{f}_i,$$

here $(\rho_1, \rho_2, \cdots, \rho_{24}) \pmod 2$ belongs to the Golay code $\mathcal{G}_{24}$.

We use $\mathcal{L}_2$ to denote the Leech lattice, and give a presentation of it. $\mathcal{L}_2$ is generated by the vectors $\pm\mathbf{f}_i \pm \mathbf{f}_j$, $1 \le i \le j \le 24$ and the vectors of the form

$$\frac{1}{2}\sum_i^{24} \rho_i \mathbf{f}_i,$$

where $(*)$ $(\rho_1, \rho_2, \cdots, \rho_{24}) \pmod 2$ belongs to the Golay code $\mathcal{G}_{24}$ and $\sum_i^{24} \rho_i \equiv 0 \pmod 4$, and

$$\mathbf{x}_0 = \frac{1}{4}(\mathbf{f}_1 + \mathbf{f}_2 + \cdots + \mathbf{f}_{23} - 3\mathbf{f}_{24}).$$

We observe that the lettice $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$ is generated by the vectors : $\pm\mathbf{f}_i \pm \mathbf{f}_j$, $1 \le i \le j \le 24$, and

$$\frac{1}{2}\sum_i^{24} \rho_i \mathbf{f}_i,$$

with the condition $(*)$.

It holds that $(\mathcal{L}_0 + \mathbb{Z}\mathbf{f}_1) \cup \mathcal{L}_0 = \mathcal{L}_1$ and $(\mathcal{L}_0 + \mathbb{Z}\mathbf{x}_0) \cup \mathcal{L}_0 = \mathcal{L}_2$. From these we get

$$\vartheta(\tau, \mathcal{L}_1) = \vartheta(\tau, \mathcal{L}_0) + \vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{f}_1),$$

$$\vartheta(\tau, \mathcal{L}_2 = \vartheta(\tau, \mathcal{L}_0) + \vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{x}_0).$$

We look at $\vartheta(\tau, \mathcal{L}_0), \vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{f}_1), \vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{x}_0)$ more precisely.

$$\begin{aligned}
\vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{f}_1) &= \sum_{\mathbf{u}\in\mathcal{L}_0} e^{\pi i(\mathbf{u}+\mathbf{f}_1, \mathbf{u}+\mathbf{f}_1)\tau} \\
&= \sum_{\mathbf{u}\in\mathcal{L}_0} e^{\pi i\{(\mathbf{u},\mathbf{u})+2(\mathbf{f}_1,\mathbf{u})+2\}\tau} \\
&= e^{2\pi i\tau} \sum_{\mathbf{u}\in\mathcal{L}_0} e^{\pi i\{(\mathbf{u},\mathbf{u})+2(\mathbf{f}_1,\mathbf{u})\}\tau} \\
&= e^{2\pi i\tau} \vartheta_{\mathbf{f}_1}(\tau, z, \mathcal{L}_0)|_{z=\tau}
\end{aligned}$$

In the same way we see that

$$\vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{f}_1) = e^{2\pi i\tau} \vartheta_{\mathbf{x}_0}(\tau, z, \mathcal{L}_0)|_{z=\tau}$$

The lattice $\mathcal{L}_0$ is seen to be obtained from the Golay code $\mathcal{G}_{24}$ by Construction B.
In conclusion if theta series and Jacobi theta series for the lattice $\mathcal{L}_0$ can be computed in a systematic way, then we may compute theta series of the Leech lattice. We can expect that $\theta(\tau, \mathcal{L}_0), \theta(\tau, \mathcal{L}_0+\mathbb{Z}\mathbf{f}_1), \theta(\tau, \mathcal{L}_0+\mathbb{Z}\mathbf{x}_0)$ will be controled by the information from the code (doubly even subcode of $\mathcal{G}_{24}$ of index 2). Numerical computation shows that

$$\vartheta(\tau, \mathcal{L}_0) = 1 + 98256q^2 + 5275648q^3 + \cdots,$$

$$\vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{f}_1) = 48q + 97152q^2 + 11509568q^3 + \cdots,$$

$$\vartheta(\tau, \mathcal{L}_0 + \mathbb{Z}\mathbf{x}_0) = 98340q^2 + 11497472q^3 + \cdots.$$

Jacobi theta series of index ≥ 2 for the Leech lattice or extremal lattices in higher dimensions should be analyzed along this direction. At present the theory of Jacobi theta series for non-unimodular lattices is not at hand.

6 Third Approach

Since there is a map from codes over $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ to the even unimodular lattices only use the construction A, there may arise a natural map from a class of codes over $\mathbb{Z}_4$ to the class of extremal theta series. We have not yet explored this direction.

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